MATH4210: Financial Mathematics Tutorial 1

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Definition (Normal Distribution)

Given a real-valued random variable $X : \Omega \to \mathbb{R}$, it follows the normal distribution with parameters μ, σ if the probability density function (pdf) of X is given by $f : \mathbb{R} \to \mathbb{R}_+$:

$$\forall x \in \mathbb{R}, f(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

It always denoted as $X \sim N(\mu, \sigma^2)$.

Exercise

Given $X \sim N(\mu, \sigma^2)$, compute the following:
(a). $\mathbb{E}(X), \mathbb{E}(X^2), Var(X)$;
(b). $\mathbb{E}(|X|), \mathbb{E}((X-K)^+)$ with K fixed;
(c). $\mathbb{E}(e^{itX})$ for t fixed (Characteristic Function).

Note that $f(t) := \mathbb{E}(e^{tX})$ is called the Moment Generating Function.

Solution:

(a). $\mathbb{E}(X) = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$

Let $y = x-\mu$. $dy = dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} (y+\mu) e^{-\frac{y^2}{2\sigma^2}} dy.$

$= \frac{1}{\sigma\sqrt{2\pi}} \left[\int_{\mathbb{R}} y e^{-\frac{y^2}{2\sigma^2}} dy + \mu \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy. \right]$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} y e^{\frac{-y^2}{2\sigma^2}} dy + \mu \cdot \int_{\mathbb{R}} f(y) dy$

$= 0 + \mu,$

$\forall a \in \mathbb{R}_+, \int_0^a y e^{-\frac{y^2}{2\sigma^2}} dy.$

$\mathbb{E}(X) = \mu.$

$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$

Change of variable, Integrate by Part

(b). Option strike K. (Vanilla).

payoff function. $f: x \mapsto (x-K)^+$

e.g. $\mathbb{E}[(S_T-K)^+ | \sigma(S_t)_{t\in[0,T]}]$

$X \sim N(\mu, \sigma^2)$

$$\mathbb{E}((X-K)^+) = \mathbb{E}((X-K)\mathbb{1}_{\{X-K>0\}})$$

$$= \underline{\mathbb{E}(X\mathbb{1}_{\{X>K\}})} - K\cdot \underbrace{\mathbb{E}(\mathbb{1}_{\{X\geq K\}})}_{=\mathbb{P}(X\geq K)}.$$

$\mathbb{E}(X\mathbb{1}_{\{X>K\}})$

$$= \int_{\mathbb{R}} x\cdot \mathbb{1}_{\{x>K\}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_K^{+\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Let $y = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma dy.$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{\frac{K-\mu}{\sigma}}^{+\infty} \cancel{\sigma}(\sigma y+\mu) e^{-\frac{y^2}{2}} dy$$

Note that $\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$ is the pdf of $N(0,1)$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{K-\mu}{\sigma}}^{+\infty} \sigma y e^{-\frac{y^2}{2}} dy + \mu\cdot \underline{\int_{\frac{K-\mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}.$$

Denote Φ is the cdf of $N(0,1)$.

$$= \underline{\frac{\sigma}{\sqrt{2\pi}} \int_{\frac{K-\mu}{\sigma}}^{+\infty} y e^{-\frac{y^2}{2}} dy} + \mu\left(1-\Phi\left(\frac{K-\mu}{\sigma}\right)\right).$$

$$= \frac{\sigma}{\sqrt{2\pi}} \cdot \left[-e^{-\frac{y^2}{2}}\right]_{\frac{K-\mu}{\sigma}}^{+\infty} + \mu\left(1-\Phi\left(\frac{K-\mu}{\sigma}\right)\right).$$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(K-\mu)^2}{2\sigma^2}} + \mu\left(1-\Phi\left(\frac{K-\mu}{\sigma}\right)\right)$$

$\mathbb{P}(X \geq K)$

Say $Y \sim N(0,1)$. then $X = \sigma Y + \mu$.

$\mathbb{P}(X \geq K) = \mathbb{P}(\sigma Y + \mu \geq K)$

$= \mathbb{P}(Y \geq \frac{K-\mu}{\sigma})$

$= 1 - \mathbb{P}(Y \leq \frac{K-\mu}{\sigma})$

$= 1 - \Phi(\frac{K-\mu}{\sigma})$

So $\mathbb{E}((X-K)^+) = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(K-\mu)^2}{2\sigma^2}} + (\mu - K)(1 - \Phi(\frac{K-\mu}{\sigma}))$

$= \sigma(\frac{1}{\sqrt{2\pi}} e^{-\frac{(K-\mu)^2}{2\sigma^2}} + \frac{\mu-K}{\sigma} \Phi(\frac{\mu-K}{\sigma}))$

$\mathbb{E}(|X|) = \mathbb{E}(X \mathbb{1}_{\{X \geq 0\}} - X \mathbb{1}_{\{X<0\}})$.

$= \mathbb{E}(X \mathbb{1}_{\{X \geq 0\}}) - \mathbb{E}(X \mathbb{1}_{\{X<0\}})$

$= \int_0^{+\infty} x \cdots - \int_{-\infty}^{0} \cdots$

(c). $\mathbb{E}(e^{itX})$ for $t \in \mathbb{R}$ given.

$t \mapsto \mathbb{E}(e^{itX})$ is called characteristic function.

$\mathbb{E}(e^{itX}) = \int_{\mathbb{R}} e^{itx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$.

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}((x-\mu)^2 - 2\sigma^2 itx)} dx$

$$(x-\mu)^2 - 2\sigma^2 i t x = x^2 - 2(\mu + \sigma^2 i t) x + (\mu + \sigma^2 i t)^2 + \mu^2 - (\mu + \sigma^2 i t)^2$$

$$= (x - (\mu + \sigma^2 i t))^2 - 2\mu\sigma^2 i t + \sigma^4 t^2$$

$$\Rightarrow \mathbb{E}(e^{itX}) = e^{\mu i t - \frac{1}{2}\sigma^2 t^2} \int_{\mathbb{R}} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 i t))^2}}_{\text{pdf of some normal.}} dx$$

$$= e^{\mu i t - \frac{1}{2}\sigma^2 t^2}$$

Exercise

Suppose $X_k \sim N\left(\mu_k, \sigma_k^2\right)$, μ_k, σ_k convergent, and $X_k \to X$ in $\mathbb{L}^2$. Show X is a normal random variable with $\mathbb{E}[X] = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

Solution:

We claim that $X_k \to X$ in $\mathbb{L}^2$ implies $X_k \to X$ in $\mathbb{L}^1$. We accept this for now.

Fix $t \in \mathbb{R}$ and $k \in \mathbb{N}$, consider the characteristic function:

$$\begin{aligned} \mathbb{E}\left[|e^{itX_k} - e^{itX}|^2\right] &\leq t^2 \mathbb{E}\left[|X_k - X|^2\right] \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

It follows from the fact: $\forall x, y \in \mathbb{R}, a \in \mathbb{R}$:

$$|e^{iay} - e^{iax}| = |\int_x^y iae^{ias} ds| \leq |a| \int_x^y 1 ds \leq |a||y - x|$$

Therefore, $e^{itX_k} \to e^{itX}$ in $\mathbb{L}^2$. By the claim, $e^{itX_k} \to e^{itX}$ in $\mathbb{L}^1$. Then

$$\mathbb{E}(e^{itX_k}) - \mathbb{E}(e^{itX}) \leq \mathbb{E}(|e^{itX_k} - e^{itX}|) \to 0$$

$$\mathbb{E}(e^{itX}) = \lim_{k\to+\infty} \mathbb{E}(e^{itX_k}) = \lim_{k\to+\infty} e^{i\mu_k t - \frac{1}{2}\sigma_k^2 t^2} = e^{it\lim_{k\to+\infty}\mu_k - \frac{1}{2}t^2 \lim_{k\to+\infty}\sigma_k^2}$$

Hence, X is normal as characteristic function uniquely identifies distributions. Moreover, the characteristic functions of X coincides with limit of that of X_k's. By continuity, we then deduce $\mathbb{E}(X) = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

The claim can be proven by Jensen's inequality or Cauchy-Schwarz inequality.

$$\begin{aligned}\mathbb{E}(|X_k - X|) &= \mathbb{E}(\sqrt{|X_k - X|^2}) \\ &\leq \sqrt{\mathbb{E}(|X_k - X|^2)} \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Exercise

Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables. For any $j \in \mathbb{N}, \mathbb{P}(Y_j = \pm 1) = \frac{1}{2}$. Define for $n \in \mathbb{N}, X_n = \sum_{j=1}^{n} Y_j$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Solution:
In order to prove that (X_n) is a martingale, we are going to verify by definition.
1. Fix $n \in \mathbb{N}$.

$$
\begin{aligned}
\mathbb{E}(|X_n|) &= \mathbb{E}(|\sum_{j=1}^{n} Y_j|) \\
&\leq \sum_{j=1}^{n} \mathbb{E}(|Y_j|) \\
&= n(1 * \frac{1}{2} + |-1| * \frac{1}{2}) \\
&= n < \infty
\end{aligned}
$$

2. Fix $n \in \mathbb{N}$, denote $\mathcal{F}_n = \sigma(X_0, ..., X_n)$.

$$\begin{aligned}\mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_n + Y_{n+1}|\mathcal{F}_n) \\ &= \mathbb{E}(X_n|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}|\mathcal{F}_n) \\ &= X_n + \mathbb{E}(Y_{n+1}) \\ &= X_n\end{aligned}$$

By 1 and 2, (X_n) is a martingale.

Remark

It still works when $\mathbb{P}(Y_j = 2) = \frac{1}{3}$ and $\mathbb{P}(Y_j = -1) = \frac{2}{3}$. (X_n) will still be a martingale as long as the expectation is 0 (Exercise!).